Algorithmic Game Theory

Mechanism Design: Single Parameter Environments

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Based on slides by Alexandros Voudouris

Single-item auctions

- A seller with **one item for sale**
- n **agents**

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- The utility of each agent is **quasilinear in money**:
  - If agent i loses the item, then her utility is 0
  - If agent i wins the item at price p, then her utility is $v_i - p$

Single-item auctions

- General structure of an auction:
  - **Input:** every agent i submits a bid b_i (*agents* = *bidders*)
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  - **Truthful auctions** that **maximize the social welfare**

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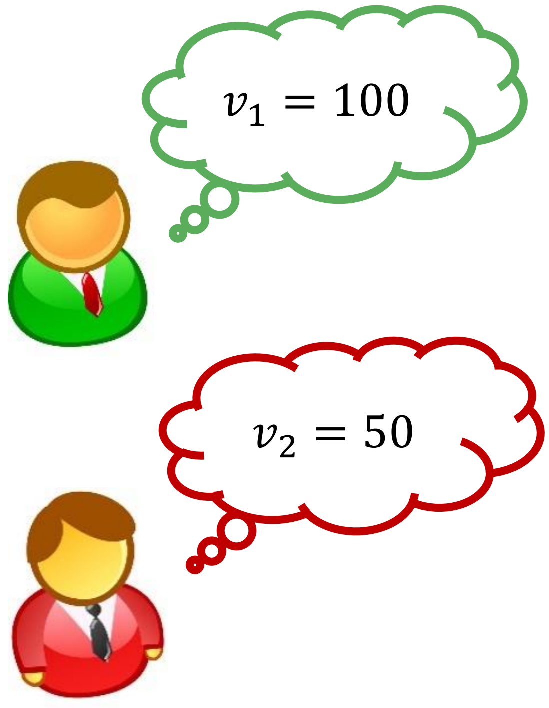

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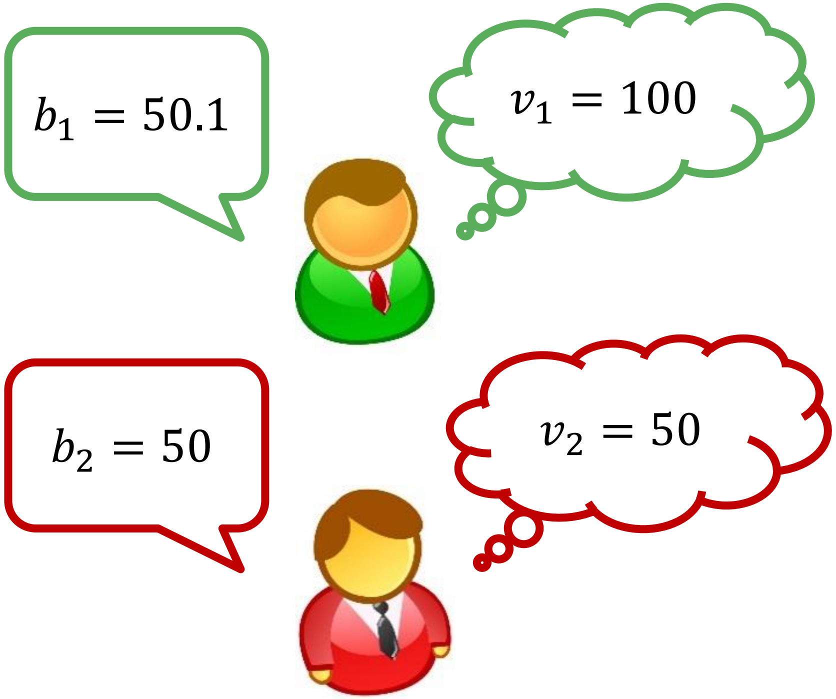

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Theorem [Vickrey, 1961]
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- (b) is obvious:
  - the selling price is at most the winner's bid, and the bid of a truthtelling bidder is equal to her true value

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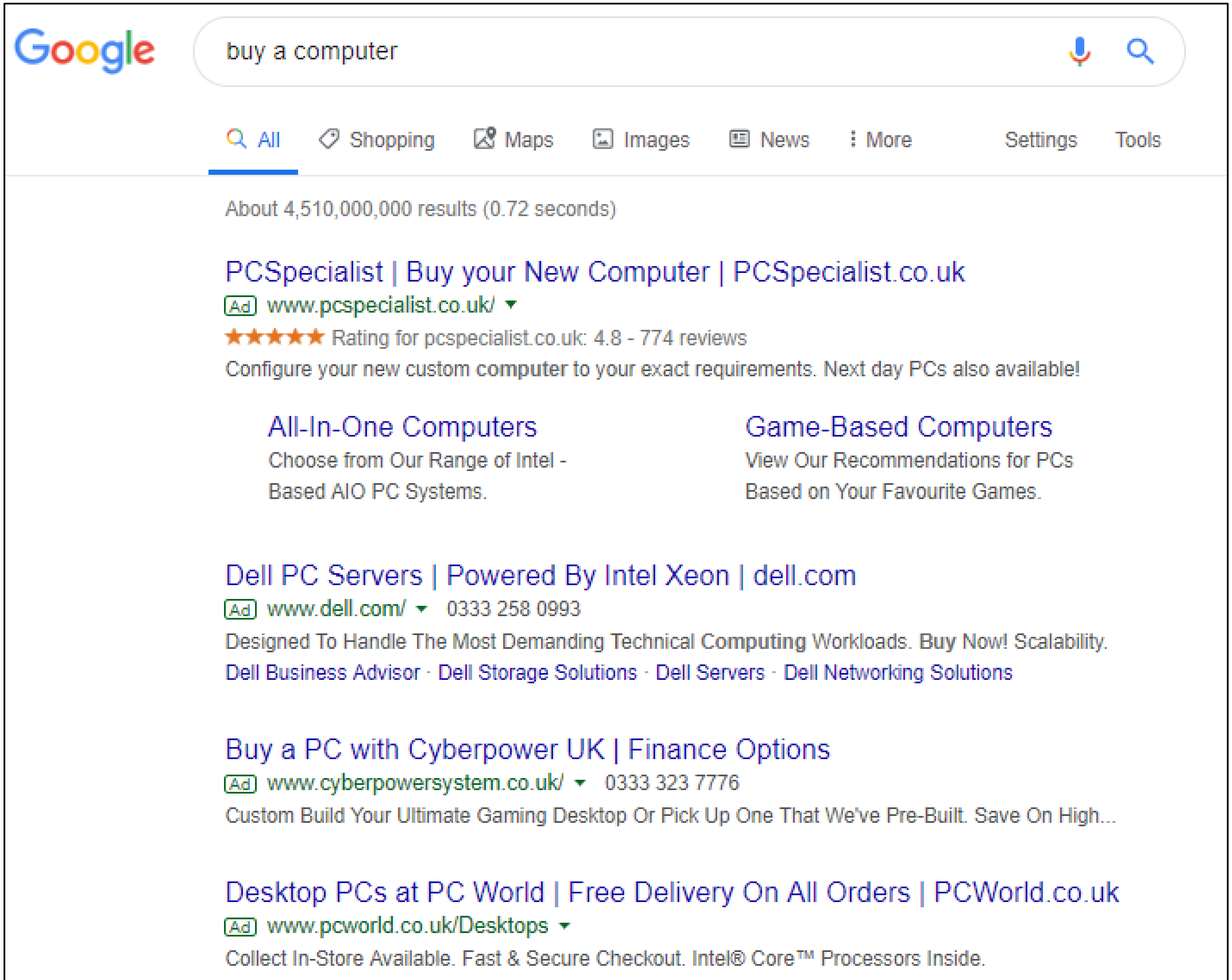
Case II: $\boldsymbol{v_i \geq B}$

- Maximum possible utility $= v_i - B$
- Bidder i wins the item by setting $b_i = v_i$ $\square$

Sponsored search auctions



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- The slots are ranked so that $a_1 \geq \cdots \geq a_k$
- Each bidder i has a **private value** v_i **per click**
  - Bidder i derives utility $a_j \cdot v_i$ from slot j

Sponsored search auctions: goals

- **Truthfulness:** It is a dominant strategy for each bidder to bid her true value
- **Social welfare maximization:** $\sum_i v_i \cdot x_i$
  - x_i is the CTR of the slot that bidder i is assigned to, or 0 otherwise
- **Poly-time execution:** running the auction should be quick

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- Can we extend the ideas we exploited for single-item auctions?

Generalized second-price auction

- **Allocation rule:** sort the bidders in decreasing order of their bids and rename them so that $b_1 \geq \cdots \geq \ b_n$
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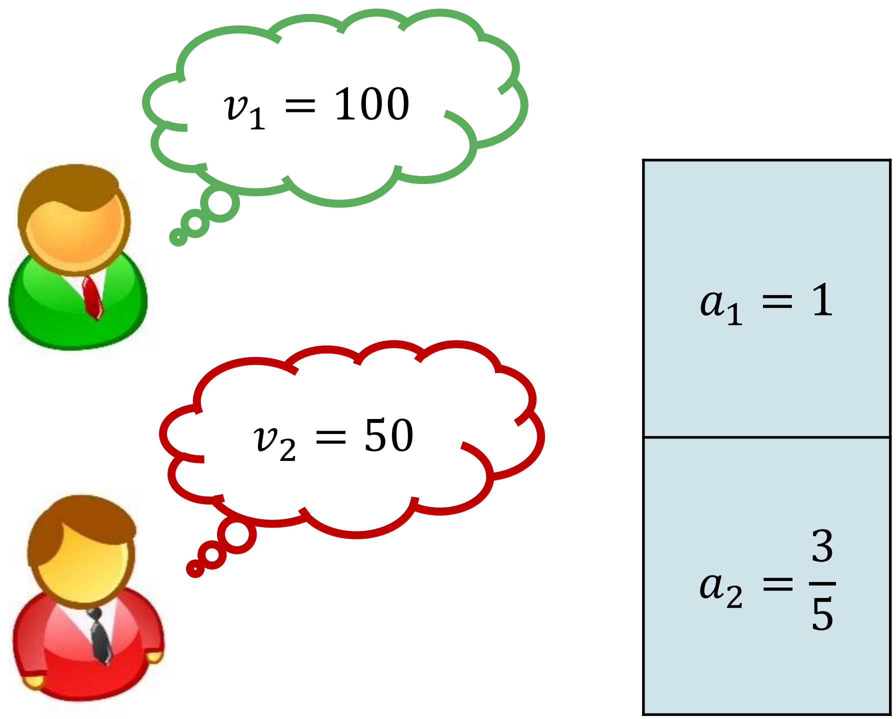

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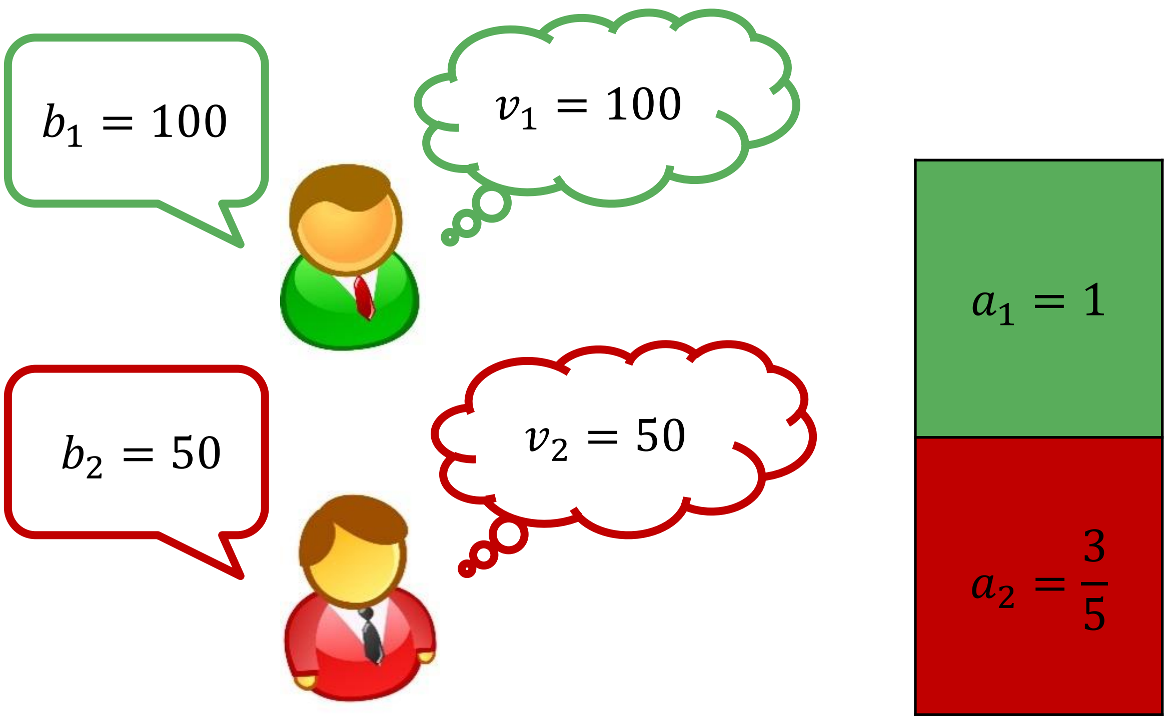

$$u_1 = 1 \cdot (100 - 50) \\ = 50$$

$$u_2 = \frac{3}{5} \cdot 50 = 30$$

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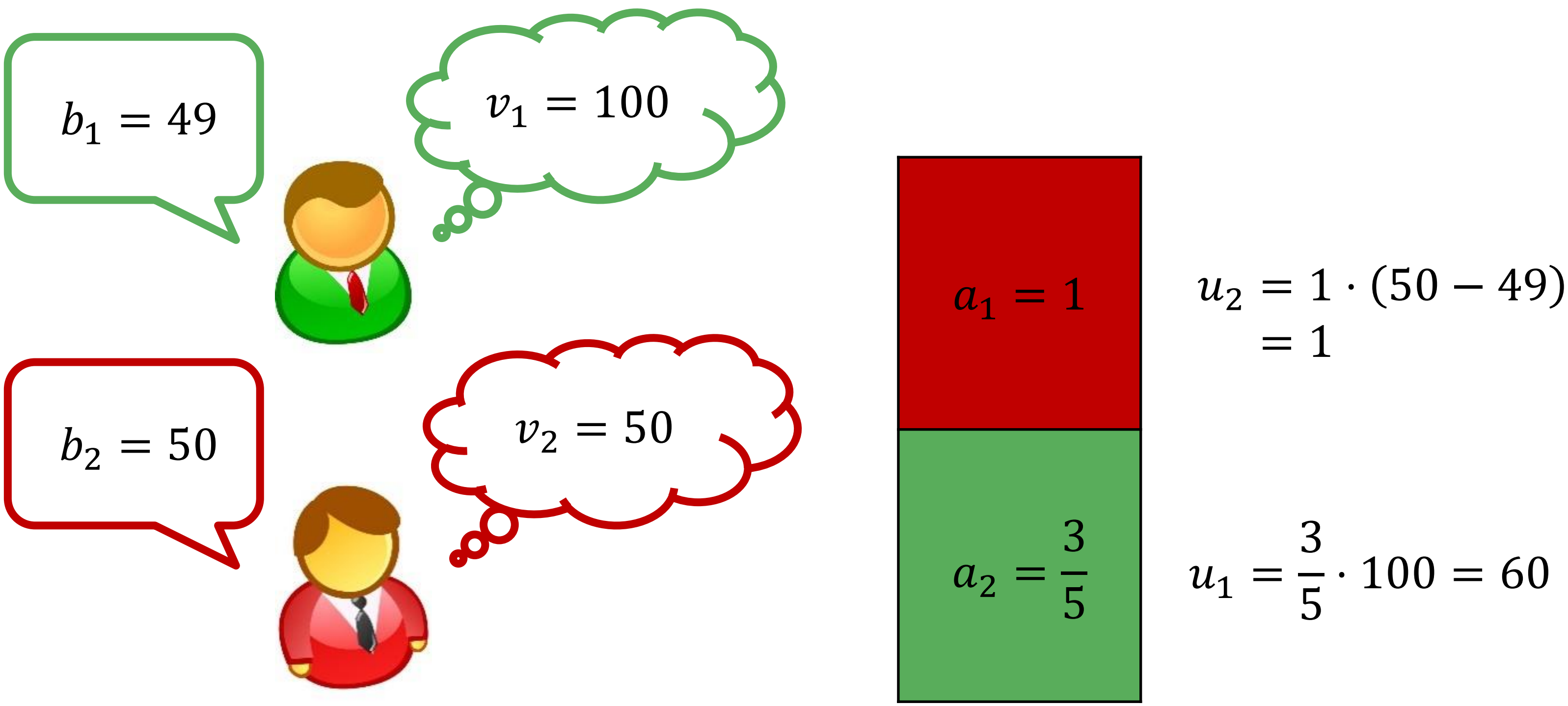

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- Focus on payment rules such that $p_i(\boldsymbol{b}) \in [0, b_i \cdot x_i(\boldsymbol{b})]$
  - $p_i(\boldsymbol{b}) \geq 0$ ensures that the seller does not pay the bidders
  - $p_i(\boldsymbol{b}) \leq b_i \cdot x_i(\boldsymbol{b})$ ensures non-negative utility for truthful bidders

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<u>Lemma</u> [Myerson, 1981]

(a) An allocation rule $\boldsymbol{x}$ is implementable if and only if it is monotone

(b) For every allocation rule $\boldsymbol{x}$, there exists a unique payment rule $\boldsymbol{p}$ such that $(\boldsymbol{x}, \boldsymbol{p})$ is a truthful auction

Proof of Myerson's Lemma

- Fix a bidder i, and the bids $\boldsymbol{b}_{-i}$ of the other bidders
- Given that these quantities are now fixed, we simplify our notation:
  - $x(z) = x_i(z, \boldsymbol{b}_{-i})$
  - $p(z) = p_i(z, \boldsymbol{b}_{-i})$
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- The idea:
  - assuming $(\boldsymbol{x}, \boldsymbol{p})$ is a truthful auction, the bidder has no incentive to unilaterally deviate to any other bid
  - This will give us a relation between $\boldsymbol{x}$ and $\boldsymbol{p}$, which we can use to derive an explicit formula for $\boldsymbol{p}$ as a function of $\boldsymbol{x}$

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  ⇨ (a) is now proved

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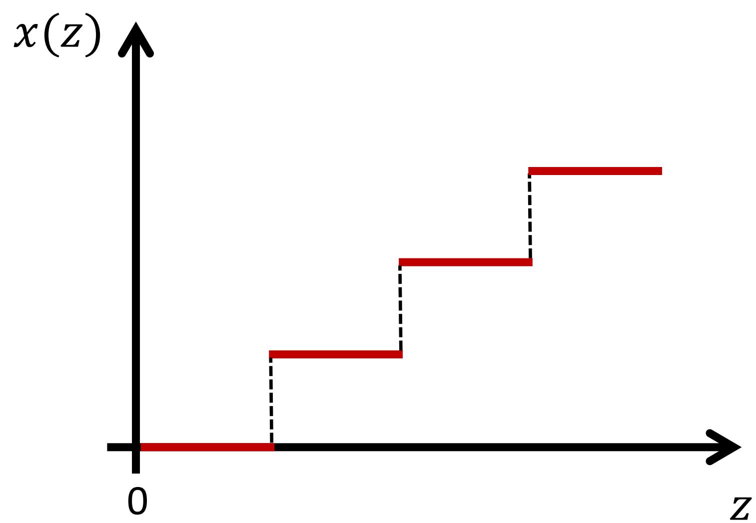
- Assume $\boldsymbol{x}$ is piecewise constant, like in sponsored search auctions



- The break points are defined by the highest bids of the other bidders

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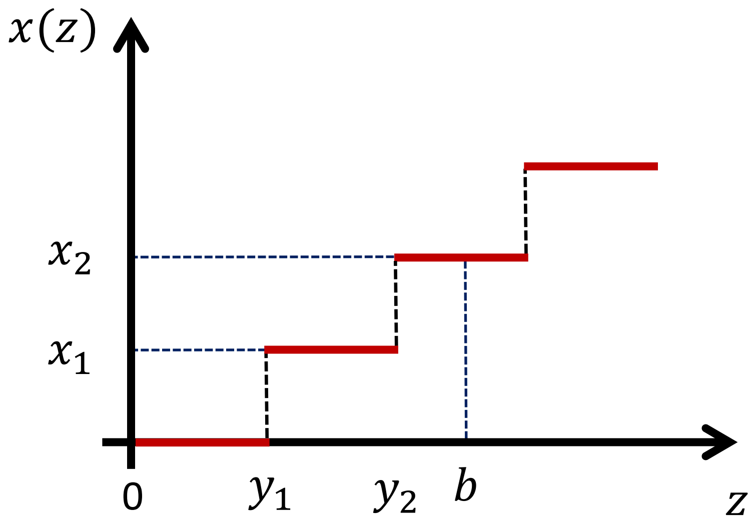
- Therefore, we can define the payment of the bidder as

$$p(b) = \sum_{y \in [0,b]} y \cdot (\text{jump of } x \text{ at } y)$$

where y enumerates all break points of x in $[0, b]$

Proof of Myerson's Lemma

- Example:



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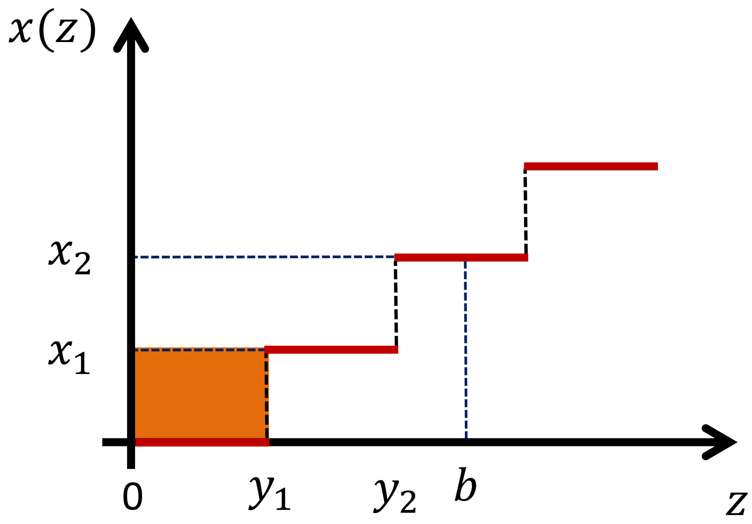

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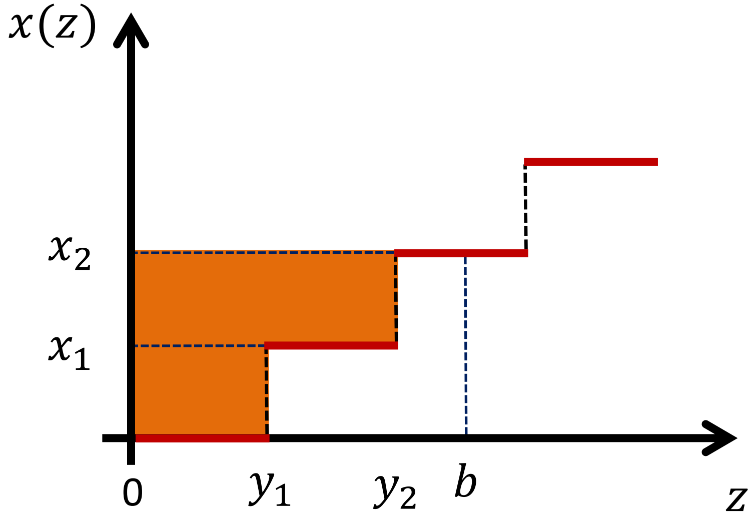

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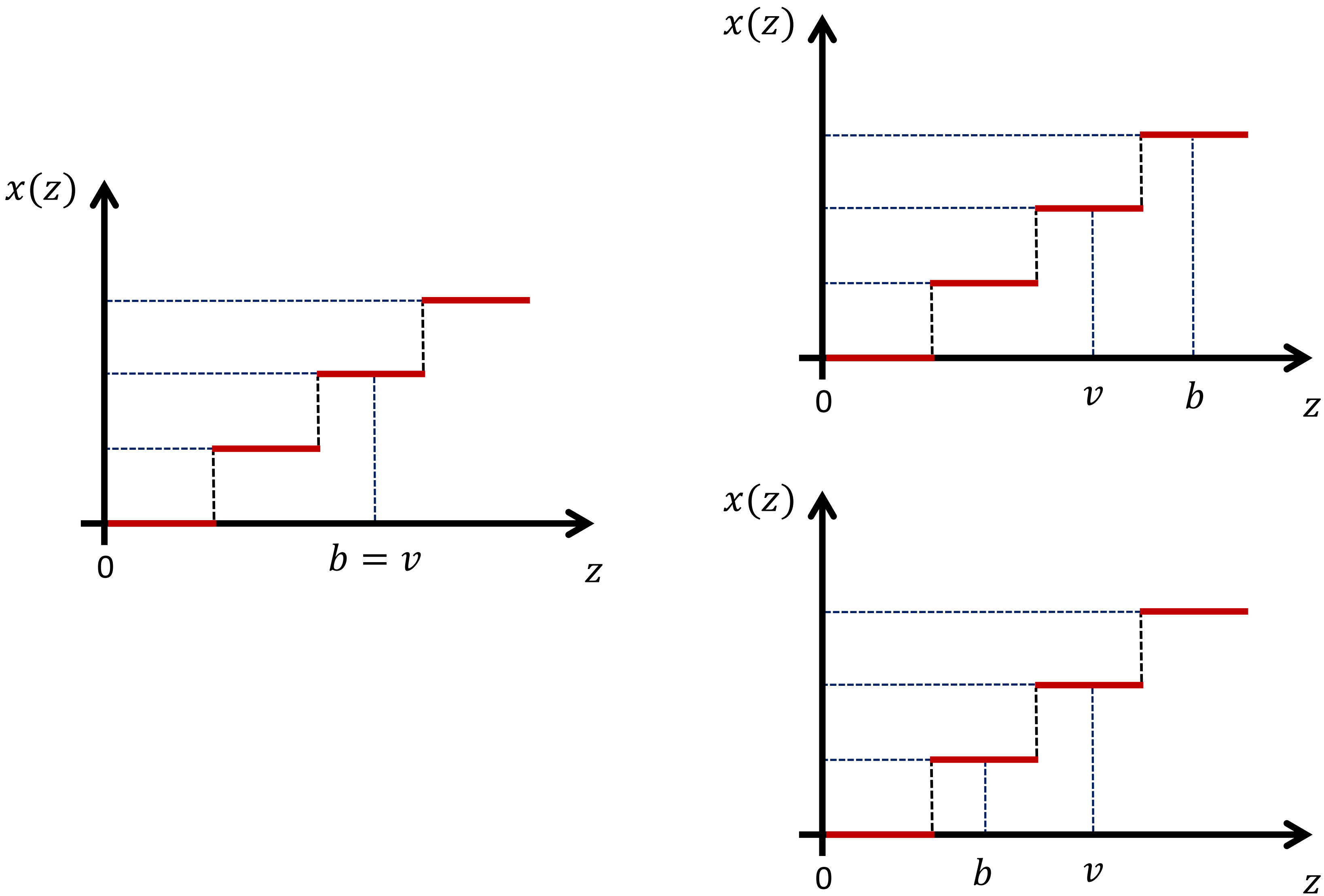

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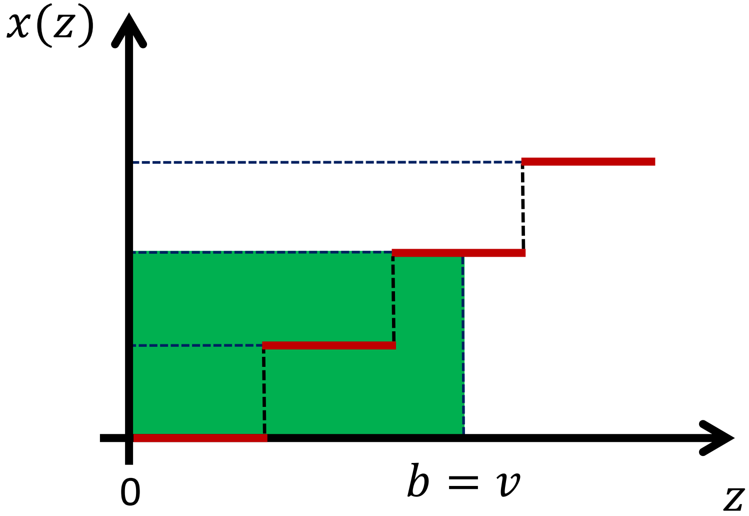

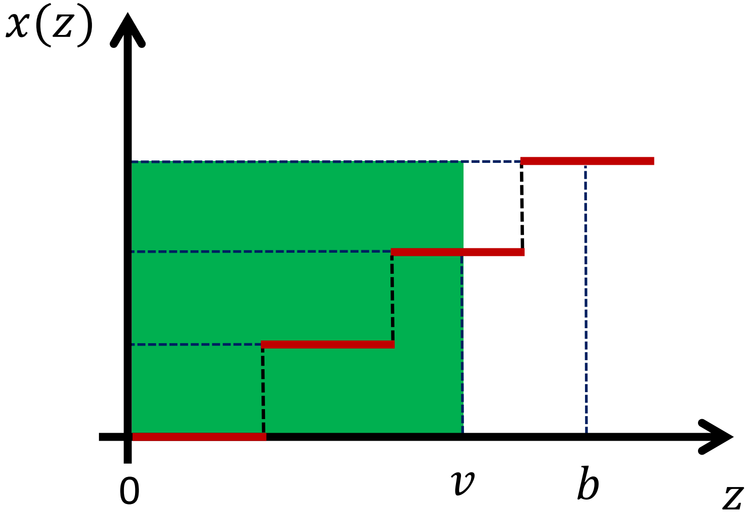

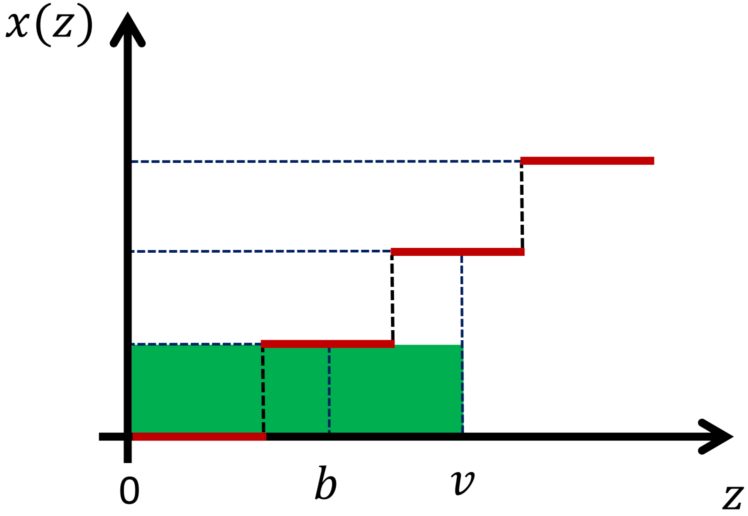

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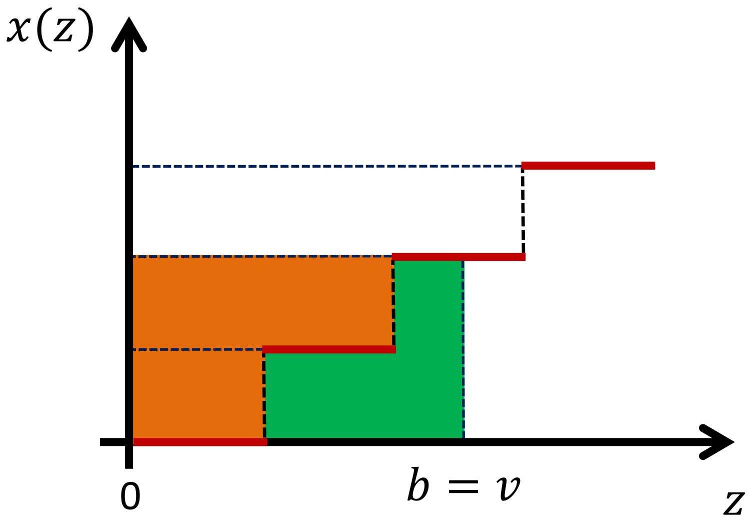

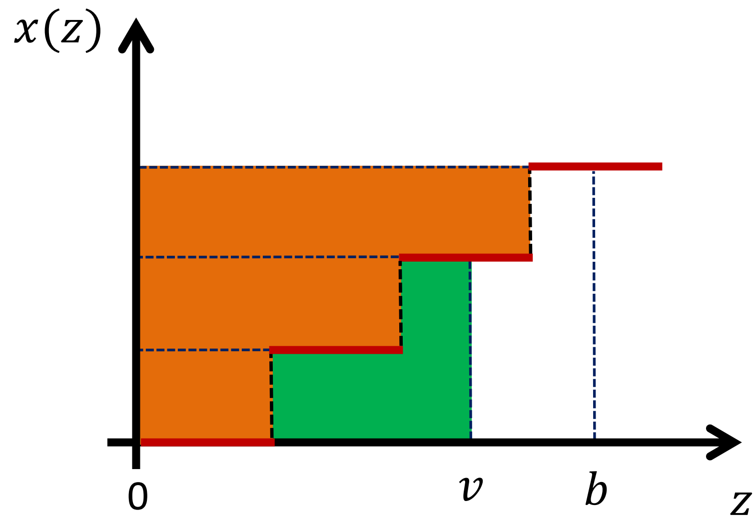

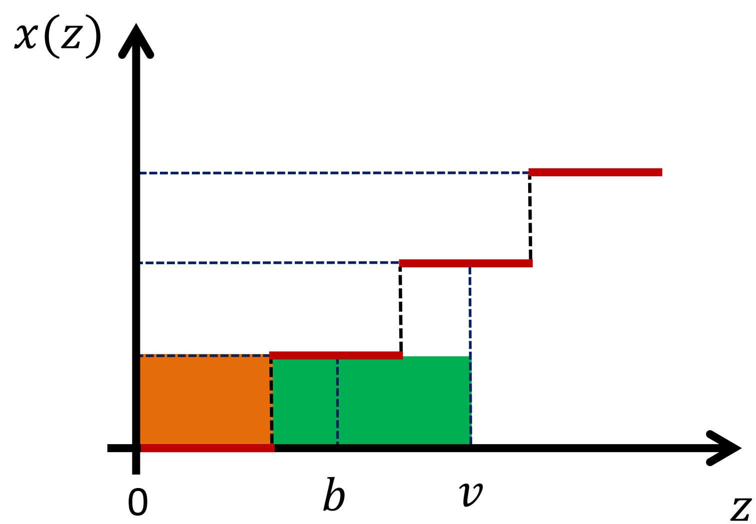

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  - In other words, y enumerates the slots from worst to best
- jump of x at y: the difference in CTR between two consecutive slots
- The total payment of the i-th highest bidder is:

$$p_i(b_i, \boldsymbol{b}_{-i}) = \sum_{j=i}^{k} b_{j+1}(a_j - a_{j+1})$$

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